WEIERSTRASS'S ELLIPTIC INTEGRAL.

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It is proposed in the following note to apply the theory of the imaginary variable, so elaborately expounded by Briot and Bouquet,* to the definite integral by which Weierstrass's p-function is generally defined.† It is thought that the method of introducing the infinite values of the variable in connection with the formation of the elementary contours presents some novelty.

The definite integral in question is

$$u = \int_{\infty}^{y} \frac{dy}{\sqrt{4y^3 - g_2 y - g_3}} = \int_{\infty}^{y} \frac{dy}{2\sqrt{(y - e_1)(y - e_2)(y - e_3)}},$$

Weierstrass's p-function being defined by the auxiliary equation

$$y = \mathrm{p}(u).$$

The initial point of the independent variable is regarded as situated upon the circumference of a circle of infinite radius, R, and having its centre at the origin. For any such point we may write

$$y = Re^{i\theta};$$

whence

$$\sqrt{(y - e_1)(y - e_2)(y - e_3)} = R^{\frac{3}{2}} e^{\frac{3}{2} i\theta}.$$

The value of the radical which results at any point of this circumference, by taking the argument of the variable positive and less than 2π, we may call its *positive* value. The radical also admits a value differing from the preceding in sign. This we will call the *negative* value.

The integral with respect to the entire circumference, or any part of it, is zero, for we have

$$u = \frac{i}{2R^{\frac{1}{2}}} \int e^{-\frac{1}{2} i\theta} d\theta,$$

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* Théorie des Fonctions Elliptiques; Paris, 1875.

† Cf. Traité des Fonctions Elliptiques, G. H. Halphen, Vol. I, p. 55; Paris, 1886.

which, since R is infinite, is equal to zero. Hence the introduction of this circumference, or any part of it, at the beginning of the path of the variable, is equivalent simply to a change in the initial value of the radical. If two paths of the variable are identical except for the fact that one is preceded by this circumference, the corresponding values of the integral differ in algebraic sign.

The critical points of the differential function are e_1, e_2, e_3. The integral with respect to an infinitely small circumference described about any one of these points, or with respect to any part of such a circumference, is zero. For upon the circumference about one of these points, e_n, the independent variable takes the form

$$y = e_n + re^{i\theta},$$

in which r is infinitely small. Hence

$$\sqrt{(y-e_1)(y-e_2)(y-e_3)} = Er^{\frac{1}{2}}e^{\frac{1}{2}i\theta},$$

E denoting a quantity differing infinitely little from a finite constant, and

$$\int \frac{dy}{2\sqrt{(y-e_1)(y-e_2)(y-e_3)}} = \frac{ir^{\frac{1}{2}}}{2E}\int e^{\frac{1}{2}i\theta}d\theta = 0.$$

The introduction of such a circumference, or any part of it, into the path of the variable, therefore simply changes the corresponding value of the radical.

Let us now consider the value of the integral when, the limit y having any value whatever, and the initial value of the radical being taken *positive*, the path of the independent variable is the exterior segment of the infinite radius drawn from the origin through y.* If we call the corresponding value of the integral $\mathrm{P}^{-1}(y)$, we have, substituting for the variable the product yt, in which t is a real quantity varying between infinity and unity,

$$\mathrm{P}^{-1}(y) = \frac{1}{2y^{\frac{1}{2}}}\int_{\infty}^{1} \frac{dt}{\sqrt{\left[t-\frac{e_1}{y}\right]\left[t-\frac{e_2}{y}\right]\left[t-\frac{e_3}{y}\right]}}.$$

If the rectilinear path along the radius includes any of the critical points e_1, e_2, e_3, the resulting indetermination is avoided by introducing into the path of the variable at each critical point an infinitely small semi-circumference

* Cf. Hermite, Cours à la Sorbonne, Quatrième édition, p. 76.

described about it in a positive direction as the variable moves toward the origin. To fix the significance of $P^{-1}(0)$, that is when the limit y is at the origin, the path of the variable is taken along that radius which passes through real and positive quantities. Under these suppositions $P^{-1}(y)$ has for every value of y a single determinate value. This value of the integral is not however obtainable solely by means of the path described above. Applying Cauchy's theorem upon definite integrals, we see that any two paths which lead from infinity to the same value of y, and which enclose no critical point between them, furnish the same value of the integral.

Let us now consider the value of the integral for an elementary contour, that is for a path of the variable which begins at the outer extremity of an infinite radius through one of the critical points and leads inward to a point infinitely near the critical point, around the circumference of an infinitely small circumference enclosing the critical point, and thereupon back to infinity along the path of approach. We thus obtain the three integrals,

$$2P^{-1}(e_1) = e_1^{-\frac{1}{2}} \int_{\infty}^{1} \frac{dt}{\sqrt{\left[t-1\right]\left[t-\frac{e_2}{e_1}\right]\left[t-\frac{e_3}{e_1}\right]}} = 2\omega_1,$$

$$2P^{-1}(e_2) = e_2^{-\frac{1}{2}} \int_{\infty}^{1} \frac{dt}{\sqrt{\left[t-\frac{e_1}{e_2}\right]\left[t-1\right]\left[t-\frac{e_3}{e_2}\right]}} = 2\omega_2,$$

$$2P^{-1}(e_3) = e_3^{-\frac{1}{2}} \int_{\infty}^{1} \frac{dt}{\sqrt{\left[t-\frac{e_1}{e_3}\right]\left[t-\frac{e_2}{e_3}\right]\left[t-1\right]}} = 2\omega_3.$$

By means of the theorem of Cauchy the most general path of the independent variable may be reduced to a combination of arcs of the infinite circumference and elementary contours, followed by a rectilinear path along the exterior segment of the infinite radius from the origin through the point y. The most general value of the integral for a given value of y consequently will be

$$2m_1\omega_1 + 2m_2\omega_2 + 2m_3\omega_3 \pm P^{-1}(y),$$

where m_1, m_2, m_3 are any integers positive or negative. From this we see that the p-function is a periodic function satisfying the equation

$$p\,(2m_1\omega_1 + 2m_2\omega_2 + 2m_3\omega_3 \pm u) = p\,(u).$$

The periods $2\omega_1$, $2\omega_2$, $2\omega_3$ are not independent of one another. A path consisting of the infinite circumference described in the positive direction, in combination with all three of the elementary contours successively described in the negative direction, is equivalent to a closed path situated in the finite region of the plane and enclosing no critical points (Fig. 1). Hence the corresponding value of the integral is zero, and we have, when e_1, e_2, e_3 are in the order of increasing arguments, the relation

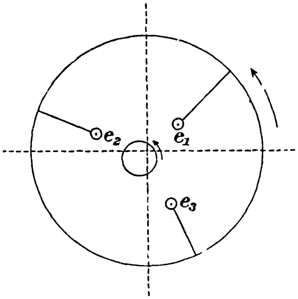

Fig. 1.

$$2\omega_1 - 2\omega_2 + 2\omega_3 = 0.$$

When the coefficients of the cubic

$$4y^3 - g_2 y - g_3$$

are real, the quantities e_1, e_2, e_3 may be all real, or one may be real and two conjugate imaginaries. In the first case the discriminant, $g_2^3 - 27g_3^2$, is positive; in the second case it is negative.

In the first case* the three quantities e_1, e_2, e_3 are all situated on the same diameter of the infinite circle, viz. the axis of reals. Let them be chosen so as to satisfy the condition $e_1 > e_2 > e_3$ (Figs. 2, 3). Of the two integrals,

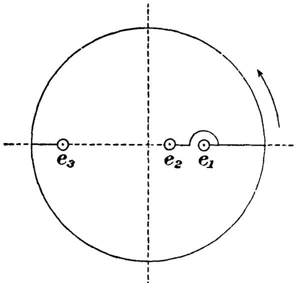

Fig. 2.

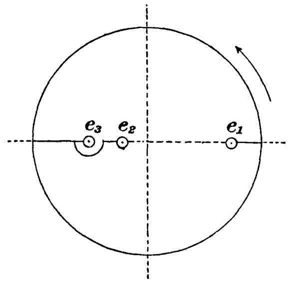

Fig. 3.

$$2\omega_1 = \int_{\infty}^{e_1} \frac{dy}{\sqrt{(y-e_1)(y-e_2)(y-e_3)}}, \qquad 2\omega_3 = \int_{-\infty}^{e_3} \frac{dy}{\sqrt{(y-e_1)(y-e_2)(y-e_3)}},$$

in each of which the independent variable is real, the former is real and the latter is a pure imaginary. From Fig. 2 we obtain, as above, $2\omega_1 - 2\omega_2 + 2\omega_3 = 0$, or $\omega_2 = \omega_1 + \omega_3$; in Fig. 3, however, this relation becomes $2\omega_1 - 2\omega_3 + 2\omega_2 = 0$, or $-\omega_2 = \omega_1 - \omega_3$.

* Cf. Halphen, Vol. I, pp. 55-57.

In the second case* let us suppose first that e_2 is real and positive (Fig. 4). ω_2 will be real. If now we take as an initial point negative real infinity, choosing the *positive* value of the radical, the quantity $2\omega_1 + 2\omega_3$ may be obtained not only from the path consisting of the infinite circumference described in a positive direction in connection with the elementary contours about e_1 and e_3, but also from the path consisting of the left-hand segment of the axis of reals taken twice but in opposite directions, and the infinitely small circumference enclosing e_2. The quantity

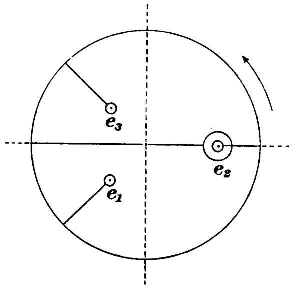

Fig. 4.

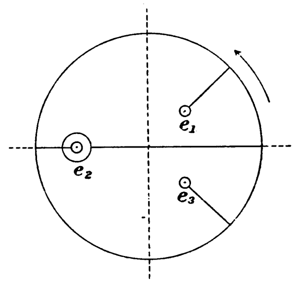

Fig. 5.

$\omega_1 + \omega_3 = \omega'_2$, which is a semi-period independent of ω_2, is thus seen to be a pure imaginary. If, on the contrary, we suppose e_2 is negative (Fig. 5), we have ω_2 a pure imaginary, and $\omega_1 - \omega_3 = \omega'_2$, which results from the path formed by the right-hand segment of the axis of reals, a real quantity. We may write then under either supposition for the real and imaginary periods respectively,

$$2\omega = \int_{\infty}^{e_2} \frac{dy}{\sqrt{(y-e_1)(y-e_2)(y-e_3)}}, \qquad 2\omega' = \int_{-\infty}^{e_2} \frac{dy}{\sqrt{(y-e_1)(y-e_2)(y-e_3)}},$$

in each of which the independent variable is real.

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* Cf. Halphen, Vol. I, pp. 69-72.